“Pseudo Random” Generators within Cryptographic Algorithms

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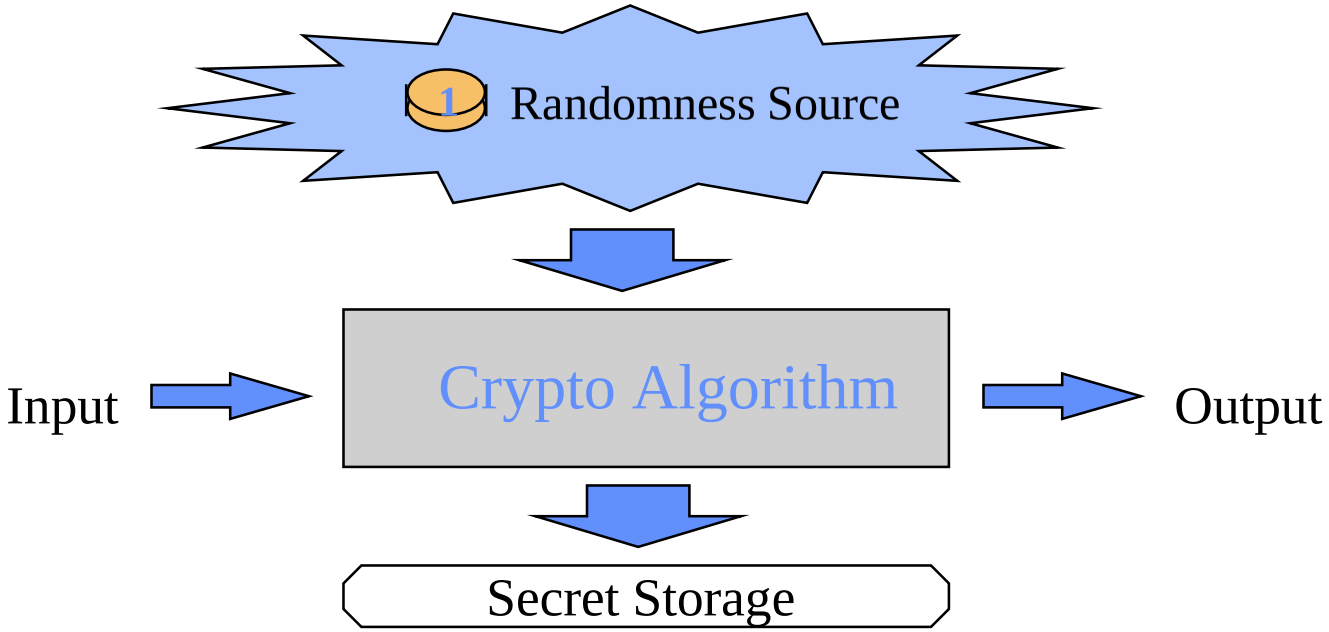
Randomness in Cryptographic Algorithms



Secret Keys, **Integral Part of Algorithm**

Randomness in Crypto Example (1)

Randmized Encryption

- Probabilistic Encryption [GM82, AD97]
- RSA'(m)=RSA(m+Random Pad) [BR]

  If randomness used for message is revealed, can decrypt message
- Randomness is necessary for Partial Information Security

Necessary for Partial-Information Security

Randomness in Crypto Example (2)

Randomized Signatures

- Signatures provably secure against chosen-message-attack [GMRi84,NY,DN]

  Random Choices are Revealed

- Digital Signature Standard [ElGamal, DSA]

  If random choices for 1 message are revealed then secret key is compromized

Source of Randomness

- Specialized Hardware exploiting Some Physical process
- User Input: Every time random number used, user is queried
- Pseudo Random Number Generators:

  Deterministic Programs that stretch a “truly random” seed into a longer sequence of “seemingly random” bits

Pseudo-Random Generation (PSRG)

- Cryptographically Strong PSRG:
  - Cannot *predict* b_i from $b_1 ... b_{i-1}$, in polynomial time
  - If there exists one-way functions, they exist
  - computationally expensive often
- Cryptographically "Not Strong'' PSRG:
  - Often pass many statistical tests
  - Often, efficient prediction algorithm exists, or can hardly analyze tests at all
  - Much faster than strong ones

⇒ Thus, often used anyway and the security depends on the cryptographic application

Our Work

- PSRG usually studied as stand alone algorithms

  (Statistical properties, prediction properties).
- We combine study of cryptographic algorithm and the PSRG used

  We analyze the use of linear congruential generators (LCG) within the DSS signing algorithm and show that it is insecure for a wide class of LCG's

[Ba] Proved Rabin-Miller primality testing with LCG

[Kr] Prove security of MAC's using LFSR (any e-bias seq's)

Results

- DSS is totally breakable when used with LCG
- DSS is totally breakable when used with truncated LCG or concatenated LCG, i.e. LCG's for which for security reasons only a fraction of the bits are used
- In general: any PSRG that can be expressed by modular linear equations in insecure when used with DSS

Digital Signature Algorithm (DSS, variant of EG)

- Let p,q primes |p| = 512, |q| = 160,
  g generator of subgroup Zp* of order q
- To sign **m** in Zq: Hash m
  Pick at random "nonce" **k.**
  Compute $\mathbf{r = (g^k \bmod p) \bmod q}$*
  $\mathbf{s = (xr+m)\, k^{-1} \bmod q}$
  Output **(r,s)**
- To verify **(r,s,m):**
  Check that $\mathbf{r = (y^{r/s} g^{m/s} \bmod p) \bmod q}$

public key	secret key
p,g,q,y	x in Zq

Where $y=g^x \bmod p$

* Signing is cheaper than Verifying, if choices of nonces is done offline which is recommended.

Security Properties and Features

– If discrete log is easy then totally breakable

– Existential forgery with key-only attack,

  Universal forgery with chosen signature attack

– If for a single message the nonce k is found, then the secret key x can be found

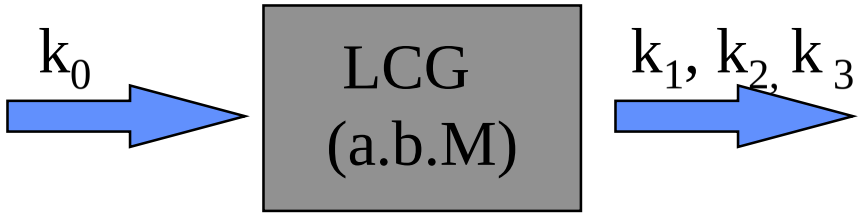
  - where do you store the off-line generated nonces for on-line usage?
  - how do you ensure they are never reused?
  - how do you generate the nonces: DSS calls for Random or Pseudo Random

Linear Congruential Generators

k_0 *truly random seed*

$k_{i+1} = a\ k_i + b \bmod M$

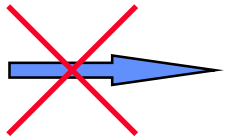
(where a,b,M define the generator)



Predictable !!!

Even if a,b,M unknown [Pl]

Even if truncated [FHLK]

However, predictability  insecurity within any crypto application as the pseudo random sequence of k_i's can be hidden

(in particular: can't use prediction algorithms)

The attack on simplest LCG

- Get the signatures of two consecutive messages $(r_1,s_1) = DSA(x, m_1, k_1)$

$$(r_2,s_2) = DSA(x, m_2, k_2)$$

Take signature equations

$$\mathbf{k_1}s_1 - \mathbf{x}r_1 = m_1 \mod q$$
$$\mathbf{k_2}s_2 - \mathbf{x}r_2 = m_2 \mod q$$
$$\mathbf{k_2} = a\mathbf{k_1} + b \mod M$$

IGNORE all relations $r = g^k \mod p$

Need to Address:
1.need to solve linear system in different moduli
2.assuming found solution, may not be meaningfull, i.e not right x,k.

1. Lattice Reduction
2. Uniqueness Lemma

The Uniqueness Lemma

Let G be any pseudo-random number generator and M a bound on the number of seeds.

Note: G is not necessarily a LCG.

Lemma:

Fix secret key x of DSS, and seed k of the generator G. Pick at random $m_1 \ldots m_n$ messages from Zq . Let (r_i,s_i) be DSS signatures for m_i with the nonces computed by G on seed k.Then

$\text{Prob}_{\boldsymbol{m}}$[there exists (x',k') s.t. $x' \neq x$ and yet (r_i,s_i) are legal signatures of m_i with the nonces computed by G on k'] $< M/(q^n)$

Recall: that in DSS messages are hashed before usage...

Lattice Reduction

- Def: Let B = $(b_1,...,b_n)$ be a set of vectors in R^n. Let the lattce L(B) = integer combinations of B.
- **C**losest Lattice **V**ector **P**roblem: Given B and vector y in R^n , find nearest vector w to y in L(B)
- Babai:(CVP): Polynomial time Approximation Algorithm which on input B and y finds w in L(B) such that $||w-y|| < f \min_v ||v-y||$ where $f = 2^{n/2}$

Solving the Equations

- We want to solve modular equations

  $\mathbf{k_1} s_1 - \mathbf{x} r_1 = m_1 \mod q$

  $\mathbf{k_2} s_2 - \mathbf{x} r_2 = m_2 \mod q$

  $\mathbf{k_2} = a\mathbf{k_1} + b \mod M$

- We set up a lattice L(B) and a vector y such that any lattice vector in L(B) close to y yields a solution (x, k_1, k_2) to the modular equations

The Lattice

- Consider the lattice generated by the columns of

$$B = \begin{vmatrix} -r_1 & s_1 & 0 & q & 0 & 0 \\ -r_2 & 0 & s_2 & 0 & q & 0 \\ 0 & -a & 1 & 0 & 0 & M \\ 1/gx & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/gk_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/gk_2 & 0 & 0 & 0 \end{vmatrix}$$

where $gx = M/2$, $gk_1 = gk_2 = q/2$ are guesses for x, k_1, k_2

- Notice that L(B) contains the vector

$$X = (m_1, m_2, b, x/gx, k_1/gk_1, k_2/gk_2)^T$$

The Target Vector

- Apply Babai's algorithm to lattice B and vector

  $Y = (m_1, m_2, b, gx/(q-gx), gk_1/(M-gk_1), gk_2/(M-gk_2))$

  and hope to find lattice vector

  $X = (m_1, m_2, b, x/gx, k_1/gk_1, k_2/gk_2)^T$

  so can recover x.
- If guesses were close then will, else try all guesses in a set which is poly(log q, log M), WHP over messages will find x' that solves the linear system and within two trials x'=x.

In general: Solving Simultaneous Modular Equations

- Previous Technique generalized to work on arbitrary linear equations in different moduli. *Where do these arise in Crypto*?
- Truncated LCG in DSS can be set up as a system of equations in different moduli,
- Thus using truncated LCG in DSS is totally breakable.

Other PSRG in DSS?

- Linear Congruential Generators with truncation
- Linear Congruential Generators with Concatenation
- Any combination of the above (see unix pseudo random number generation of 32 bits)

Open Problems

- LCG with unknown parameters a,b,M
  - Still predictable
- Generators defined by non-linear modular recurrences
  - Sequences produced by polynomial modular recurrence relations are also predictable
- Can our attack be extended to these more general PRNG?